A typical solution to 1.3.T

$\bar{x} = x$ rounded to 24 sign fig, standard rounded

(a) $x \in (0, 200\,000)$ want e_x as large as possible. give x, e_x, $\bar{x}$.

---

200 000 in binary $\simeq 2^{17} + \dots$

$$= \underbrace{1\overbrace{\cdot\ \cdot\ -\ -\ -\ -\ -}^{17}}_{\text{18 places total}}.\ \cdot\ \cdot\ \cdot\ -\ {}_2$$

To make e_x large with 24 sign fig. I should choose x to have as many binary places as possible before radix pt, Also want x to round down to $\bar{x}$.

$$e_x = x - \bar{x}$$

$$\text{let } x = 1\overbrace{0\text{———}0}^{17 \text{ zeros}}.\overbrace{000000}^{6}0\overbrace{1111111}^{300}$$

$$\Rightarrow \bar{x} = 1\overbrace{0\text{———}0}^{17 \text{ zeros}}.000\,000$$

$$\bar{x} = 2^{17}$$

$$x = 2^{17} + \underbrace{2^{-8} + 2^{-9} + \dots\ 2^{-307}}_{2^{-307}\left(\underbrace{2^{299} + 2^{298} + \dots + 1}_{(2^{300} - 1)}\right)}$$

$$x = 2^{17} + 2^{-7} - 2^{-307}$$

$$e_x = x - \bar{x} = 2^{17} + 2^{-7} - 2^{-307} - 2^{17}$$
$$= \boxed{2^{-7} - 2^{-307}}$$

Note: $|e_x| \le \frac{1}{2} b^{e-B-t} = \frac{1}{2} 2^{-6}$

$$= 2^{-7}.$$

Not highest possible

But could change x to $1\underbrace{0\cdots0}_{17}.\underbrace{0\cdots0}_{6}1\underbrace{0\cdots0}_{\text{a bazillion}}1111$

$\bar{x} = 1\underbrace{0\cdots0}_{17}$

$x - \bar{x} = 2^{-7} - 2^{-HUGE} \approx 2^{-7}.$

ⓑ Calculate ε_x

$$\varepsilon_x = \frac{e_x}{\bar{x}} = \frac{2^{-7} - 2^{-307}}{2^{17}}$$
$$= \frac{2^{-7}}{2^{17}} - \frac{2^{-307}}{2^{17}}$$
$$= \boxed{2^{-24} - 2^{-324}} \approx 2^{-24}.$$

ⓒ Maximum possible

$$|\varepsilon_x| \le \frac{1}{2} b^{-m+1} = \frac{1}{2}(2)^{-24+1} = \frac{2^{-23}}{2} = \boxed{2^{-24}}.$$

Last time we gave formulas for $e_{f(x)}$ & $\varepsilon_{f(x)}$ using 2nd degree approximation (ie. Taylor 1st degree polynomial approx. + 2nd degree Lagrange remainder.) The first term is used by computers to estimate the error.

Simpler approach that relies on a little more knowledge of f'

$\longrightarrow$ use 0th order Taylor polynomial with 1st order remainder.

$f(x)$ function

$a = \bar{x}$ in Taylor formula

$$f(x) = f(\bar{x}) + f'(c)(x - \bar{x})$$

$$\Rightarrow \quad f(x) - f(\bar{x}) = f'(c)(x - \bar{x})$$

$$\Rightarrow \boxed{e_{f(x)} = f'(c)\, e_x}$$

where c is some particular # between x & $\bar{x}$.

$$\Rightarrow |e_{f(x)}| = |f'(c)|\ |e_x|$$

So if we have a good bound on $|f'|$, then we have a good bound on $|e_{f(x)}|$.

$$\Rightarrow \varepsilon_{f(x)} = \frac{e_{f(x)}}{f(\bar{x})} = \frac{f'(c)}{f(\bar{x})} e_x$$

$$\boxed{\varepsilon_{f(x)} = \frac{f'(c)}{f(\bar{x})} \bar{x}\, \varepsilon_x}$$

again, c is a number between x & $\bar{x}$.

$$\Rightarrow |\varepsilon_{f(x)}| = \left| \frac{f'(c)\, \bar{x}}{f(\bar{x})} \right| |\varepsilon_x|$$

From this, if we have upper bounds on $|f'(c)|$, $|\bar{x}|$, and a lower bound on $|f(\bar{x})|$, that will given us a bound $\left|\frac{f'(c)\,\bar{x}}{f(\bar{x})}\right| \le K$

$$\Rightarrow |\varepsilon_{f(x)}| \le K\,|\varepsilon_x|$$

Multivariable Error formula

$f(x, y, \dots)$

$$e_{f(x)} = f'(c)\, e_x$$

1-variable.

$e_x = x - \bar{x}$

$e_y = y - \bar{y}$

$$\boxed{e_{f(x,y,\dots)} = \nabla f(\underset{(c_1, c_2, \dots)}{c}) \cdot \begin{pmatrix} e_x \\ e_y \\ \vdots \end{pmatrix}}$$

multivariable error propagation formula.

$$\boxed{\mathcal{E}_{f(x)} = \frac{f'(c)}{f(\bar{x})}\,\bar{x}\,\mathcal{E}_x}$$ 1-variable

$\Downarrow$

$$\boxed{\mathcal{E}_{f(x,y,\dots)} = \frac{1}{f(\bar{x},\bar{y},\dots)} \nabla f(c_1,c_2,\dots)\cdot\begin{pmatrix}\bar{x}\,\mathcal{E}_x\\ \bar{y}\,\mathcal{E}_y\\ \vdots\end{pmatrix}}$$

Relative error version.

where $c=(c_1,c_2,\dots)$
is a point on the line
connecting $\bar{x}$ & x.

$\Rightarrow$ with a bound on $|\nabla f(c)|$,
you can estimate the maximum $|\mathcal{E}_{f(x)}|$
$|e_{f(x)}|$

eg $\left| e_{f(x,y,\ldots)} \right| = \left| \nabla f(c_1, c_2, \ldots) \cdot \begin{pmatrix} e_x \\ e_y \\ \vdots \end{pmatrix} \right|$

$$\leq \left| \nabla f(c_1, c_2, \ldots) \right| \left| \begin{pmatrix} e_x \\ e_y \\ \vdots \end{pmatrix} \right|$$

$$\left| (v_1, v_2, v_3 \ldots) \right| = \sqrt{v_1^2 + v_2^2 + \cdots}$$

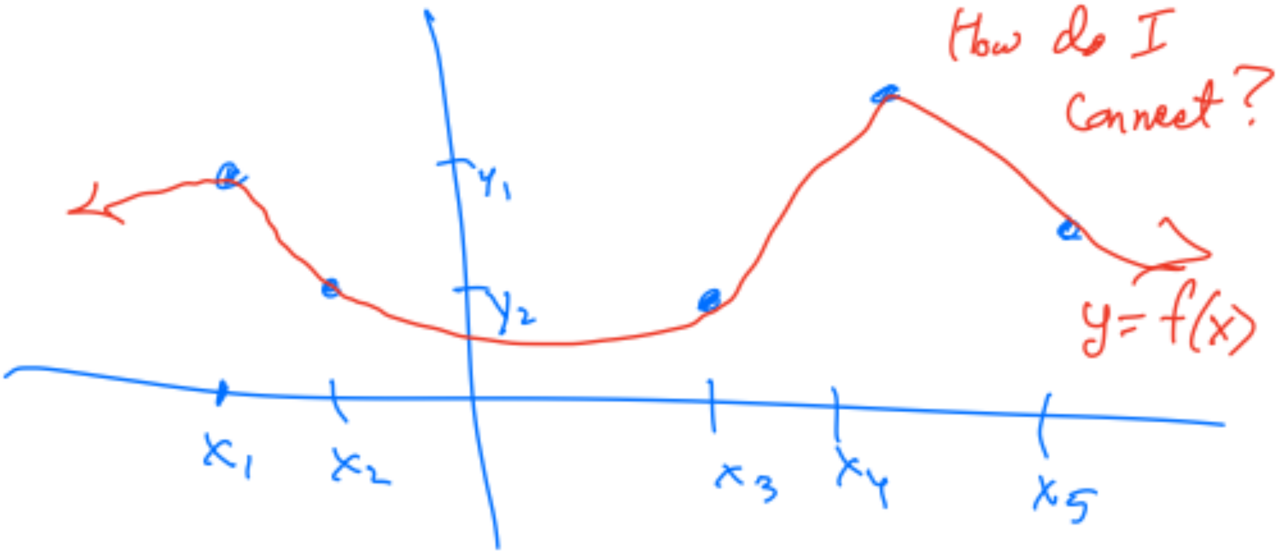
Example of Next Topic : Interpolation.

Problem: Have a bunch of data pts $(x_1, y_1), (x_2, y_2), \ldots$

How do I make a fcn. $f(x)$

s.t. $f(x_1) = y_1$

$f(x_2) = y_2$, etc.



Example 1: Connect two points.

$$y = mx + b$$

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \boxed{y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

Note also

$$y = y_2 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_2)$$

Why are they the same?

1st one $$y = \frac{y_1(x_2 - x_1)}{(x_2 - x_1)} + \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

$$y = \frac{y_1(x_2 - x_1) + (y_2 - y_1)(x - x_1)}{(x_2 - x_1)}$$

$$\Rightarrow y = \frac{y_1 x_2 - \cancel{x_1 y_1} + y_2 x - x_1 y_2 - y_1 x + \cancel{x_1 y_1}}{(x_2 - x_1)}$$

$$\Rightarrow y = \frac{y_1 x_2 + y_2 x - x_1 y_2 - y_1 x}{(x_2 - x_1)}$$

$$y = \frac{y_1(x_2 - x)}{(x_2 - x_1)} + \frac{y_2(x - x_1)}{(x_2 - x_1)}$$

$$\Rightarrow \boxed{y = \frac{y_1(x - x_2)}{(x_1 - x_2)} + \frac{y_2(x - x_1)}{(x_2 - x_1)}}$$